

# Position space of Doubly Special Relativity

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## Abstract

We consider three possible approaches to formulating coordinate transformations on position space associated with non-linear Lorentz transformations on momentum space. The first approach uses the definition of velocity and gives the standard Lorentz transformation. In the second method, we translate the behavior in momentum space into position space by means of Fourier transform. Under certain conditions, it also gives the standard Lorentz transformation on position space. The third approach investigates the covariance of the modified Klein-Gordon equation obtained from the dispersion relation.

## 1 Introduction

Doubly Special Relativity (DSR) has been investigated extensively in the last three years [1][2][3][4][5]. When quantum effect is taken into account, classical relativity is no longer sufficient to describe spacetimes. For instance, Padmanabhan[6] argued

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that combining gravity with quantum theory prevents the measurement of a single event with an accuracy better than the Planck length  $l_p$ . The Planck length/energy is obviously expected to play a role in quantum gravity. Special relativity suggests that if the scale  $l_p$  is measured in one inertial reference frame, it may be different in another observer's frame. So one faces a direct conflict when Planck scale is introduced to special relativity. To solve this paradox, DSR theories modify special relativity with two observer-independent scales. For example, Magueijo and Smolin [7] proposed a non-linear representation of Lorentz group on momentum space such that the Planck energy is left invariant. Specifically, the non-linear representation of the Lorentz group takes the form

$$W = U^{-1}LU, \quad (1)$$

where  $L$  is the ordinary Lorentz generator acting on momentum space and  $U$  is defined by

$$U \circ p_a = \frac{p_a}{1 - l_p p_0}. \quad (2)$$

Consequently, the boosts in the  $x$  direction are given by

$$p'_0 = \frac{\gamma(p_0 - vp_x)}{1 + l_p(\gamma - 1)p_0 - l_p\gamma vp_x} \quad (3)$$

$$p'_x = \frac{\gamma(p_x - vp_0)}{1 + l_p(\gamma - 1)p_0 - l_p\gamma vp_x} \quad (4)$$

$$p'_y = \frac{p_y}{1 + l_p(\gamma - 1)p_0 - l_p\gamma vp_x} \quad (5)$$

$$p'_z = \frac{p_z}{1 + l_p(\gamma - 1)p_0 - l_p\gamma vp_x}. \quad (6)$$

A general isotropic  $U$ -map discussed by Magueijo and Smolin [8] takes the form

$$(E', \mathbf{p}') = U \circ (E, \mathbf{p}) = (Ef_1(E), \mathbf{p}f_2(E)), \quad (7)$$

Note that a general invariant quantity associated with the group action Eq. (1) is [8]

$$||p||^2 \equiv \eta^{ab}U(p_a)U(p_b) \quad (8)$$

If the invariant  $||p||^2$  is identified with the square of the mass  $||p||^2 = m_0^2$ , one obtains the general isotropic dispersion relation

$$E^2 f_1^2 - p^2 f_2^2 = m_0^2, \quad (9)$$

where  $p^2 = |\mathbf{p}|^2$ . For  $U$  given in Eq. (2), we have

$$\frac{E^2 - p^2}{(1 - l_p p_0)^2} = m_0^2. \quad (10)$$

To make the Planck energy  $E_p$  invariant under the action of the Lorentz group,  $U$  must be singular at  $E_p$ . Conservation of energy and momentum and other properties concerning the modified Lorentz transformation have been explored in [8] and [9]. The main interest of this paper is to investigate the consequences of such a transformation on position space. One non-linear Lorentz transformation on position space analogous to Eq. (1) is

$$K = (U^*)^{-1} L U^*, \quad (11)$$

where  $L$  is the standard Lorentz transformation and  $U^*$  acts on position space. If  $U^*$  is taken as [10]

$$U^*(x^\mu) = x^\mu \frac{1}{t + R}, \quad (12)$$

it leads to the Fock-Lorentz transformation [11][12]

$$t' = \frac{\gamma(t - vx)}{1 - (\gamma - 1)t/R + \gamma vx/R} \quad (13)$$

$$x' = \frac{\gamma(x - vt)}{1 - (\gamma - 1)t/R + \gamma vx/R} \quad (14)$$

$$y' = \frac{y}{1 - (\gamma - 1)t/R + \gamma vx/R} \quad (15)$$

$$z' = \frac{z}{1 - (\gamma - 1)t/R + \gamma vx/R}. \quad (16)$$

However, this treatment only provides an analogy to the transformation on momentum space. An open question is what transformations on position space are compatible with the non-linear transformations on momentum space above. We shall explore this issue in three distinct ways. First, we use velocity as a link to connect the momentum space and position space. There are different proposals on the definition of velocity in DSR theories [13][14][15][16][17]. In special relativity, the group velocity  $v_g = \frac{\partial E}{\partial p}$  is equal to the boost velocity. However, in DSR theories,  $v_g$  is always mass-dependent [14]. For example, in the Magueijo-Smolín model above, the group velocity reads,

$$v_g = \frac{v\gamma}{\sqrt{2l_p m_0 \gamma + \gamma^2 + l_p^2 m_0^2}}, \quad (17)$$

where  $v$  is the boost velocity. This seems to be odd since the velocity of a particle depends on its mass. Kosiński and Maślanka [15] demand that the velocity be a property of reference frame rather than of a particular object and then, the velocity is identified with the boost velocity  $v$ , as in the case of special relativity. From this assumption and the fact that the group structure remains the same as in Einstein's theory, the authors of [15] derived the ordinary relativistic velocity law for a general DSR theory. Daszkiewicz, *et al.* [16] investigated the velocity of particles in DSR, defining velocity as the Poisson bracket of position with appropriate Hamiltonian:

$$u_0 \equiv \dot{x}_0 = [x_0, \mathcal{H}] \quad (18)$$

$$u_i \equiv \dot{x}_i = [x_i, \mathcal{H}] \quad (19)$$

where  $(u_o, u_i)$  is the four velocity of particle. They also found that the four velocities transform as standard Lorentz four vectors and the boost parameter  $\xi$  is related to velocity in exactly the same way as in the Special Relativity, i.e.  $v = \tanh \xi$ . Based on these works, we also identify the boost velocity  $v$  as the true velocity in DSR theories and derive the classical relativistic addition law. Most importantly, from the original definition  $v = dx/dt$ , we show that the standard Lorentz transformation on position space is inferred by the velocity addition law, despite the fact that the momentum transformation is non-linear.

In section 2.2, we follow Schützhold and Unruh's prescription to induce a transformation on position space. Fourier transform plays an important role in this method. By requiring some suitable conditions, we obtain the standard Lorentz transformation again.

The third method to fix the transformation law in position space is making use of the covariance requirement on the modified Klein-Gordon equation (see Eq. (61)). We show that the only linear transformations that keep Eq. (61) invariant are pure rotations, i.e., boosts are ruled out. This may indicate that the modified K-G equation is associated with a preferred inertial frame. An alternative resolution is that the coordinate transformation inferred by the modified K-G equation is not linear. Kimberly *et. al* [10] also required that  $p_a dx^a$  remain invariant and derived an energy-dependent boost in position space (See Eq. (71) and Eq. (72)). But this result contradicts the one we obtain from the covariance of the field equation. Therefore, the condition that  $p_a dx^a$  remains invariant may not be valid in DSR theories.

For simplicity, we shall work in  $1 + 1$  dimensions. There should be no difficulties to generalize our results to four dimensions.

## 2 Transformations on position space

### 2.1 Velocity addition and coordinate transformation

As discussed in the introduction, we treat the boost velocity  $v$  as the velocity measured by an inertial reference frame. We first show that the non-linear transformation (1) indicates the ordinary relativistic velocity addition law. Since each Lorentz transformation  $L$  is determined by the relative velocity  $v$ , we rewrite Eq. (1) as

$$W(v) = U^{-1}L(v)U, \quad (20)$$

We now show that the standard Lorentz transformation

$$p'_0 = \gamma(p_0 - vp_x) \quad (21)$$

$$p'_x = \gamma(p_x - vp_0). \quad (22)$$

leads to the relativistic addition law for velocities. Let  $A$ ,  $B$  and  $C$  be three inertial observers with relative velocities  $v_{AB}$ ,  $v_{BC}$  and  $v_{AC}$ . Then we have

$$L(v_{AB})L(v_{BC}) = L(v_{AC}). \quad (23)$$

Substituting Eqs. (21) and (22) into Eq. (23), we obtain the relativistic velocity addition law

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + v_{AB}v_{BC}}. \quad (24)$$

Since a non-linear transformation is a representation of the Lorentz group, the velocity relation (24) must hold for all transformations like Eq. (20). Next, we shall show that Eq. (24) implies the standard Lorentz transformation on coordinate space. Suppose that the coordinate transformation between two inertial frames is

$$x' = g(t, x) \quad (25)$$

$$t' = f(t, x) \quad (26)$$

By differentiation, we have

$$dx' = \dot{g}dt + g'dx \quad (27)$$

$$dt' = \dot{f}dt + f'dx \quad (28)$$

Here,  $\dot{g} = \partial g / \partial t$  and  $g' = \partial g / \partial x$ , etc. Hence,

$$u'_x = \frac{dx'}{dt'} = \frac{\dot{g} + g' u_x}{\dot{f} + f' u_x}. \quad (29)$$

The relative velocity  $v$  between the two frames can be read off by taking  $dx' = 0$  in Eq. (27). Therefore,

$$v = -\dot{g}/g'. \quad (30)$$

By comparing Eq. (29) with Eq. (24), we obtain the following relations

$$g' = \dot{f}, \quad \dot{g}/\dot{f} = -v, \quad f'/\dot{f} = -v \quad (31)$$

Using these relations, we rewrite Eq. (27) and Eq. (28) as

$$dx' = -v \dot{f} dt + \dot{f} dx \quad (32)$$

$$dt' = \dot{f} dt - v \dot{f} dx. \quad (33)$$

The inverse transformation thereby is

$$dx = \dot{f}^{-1} \frac{v}{1-v^2} dt' + \dot{f}^{-1} \frac{1}{1-v^2} dx' \quad (34)$$

$$dt = \dot{f}^{-1} \frac{1}{1-v^2} dt' + \dot{f}^{-1} \frac{v}{1-v^2} dx'. \quad (35)$$

On the other hand, for symmetry reasons Eqs. (34) and (35) may be obtained from Eqs. (32) and (33) by interchanging the primed and the unprimed variables and replacing  $v$  by  $-v$ , i.e.,

$$dx = v \dot{f} dt' + \dot{f} dx' \quad (36)$$

$$dt = \dot{f} dt' + v \dot{f} dx'. \quad (37)$$

By comparing coefficients, we find immediately

$$\dot{f} = \frac{1}{\sqrt{1-v^2}} = \gamma. \quad (38)$$

Therefore, all derivatives of  $f$  and  $g$  are constants, which indicates that Eqs. (25) and (26) are linear transformations. By solving Eqs. (38) and (31), we obtain the standard Lorentz transformation on coordinate space.

## 2.2 Fourier transform method

In this section, we shall use Fourier transform to translate the behavior in momentum space into position space. Schützhold and Unruh[18] suggest the following way translating a scalar field  $\phi(t, x)$  from one frame to another:

$$\phi(t, x) \rightarrow \tilde{\phi}(E, p) = \mathcal{F}\phi \quad (39)$$

$$\tilde{\phi}(E, p) \rightarrow \tilde{\phi}(E', p') \quad (40)$$

$$\phi'(t, x) = \mathcal{F}^\dagger \tilde{\phi}(E', p'), \quad (41)$$

where  $(E', p') = U \circ (E, \mathbf{p})$  is the transformation on momentum space defined by Eq. (7) and  $\mathcal{F}$  is the Fourier transform. The above prescription enables us to induce a coordinate transformation  $(t, x) \rightarrow (U^*t, U^*x)$  determined by

$$\phi'(t, x) = \phi(U^*t, U^*x). \quad (42)$$

It is not difficult to check that the Lorentz transformation can be recovered this way from the usual Lorentz transformation on momentum space. However, since we deal with non-linear transformations, it is not easy to work out a result from a general  $\phi(t, x)$ . So we shall replace  $\phi(t, x)$  with a single coordinate  $t$  or  $x$  and by following Eqs. (39)-(41), obtain the coordinate transformation directly. Hence, Eq. (39) yields

$$\mathcal{F}[t] = -i \frac{\partial}{\partial E} \delta(E) \delta(p) \quad (43)$$

$$\mathcal{F}[x] = i \frac{\partial}{\partial p} \delta(p) \delta(E). \quad (44)$$

According to Eqs. (40) and (41), we change variables from  $(E, p)$  to  $(E', p')$  and then Fourier transform them back to get new coordinates on position space. The operations on  $t$  and  $x$  respectively give rise to the new coordinates:

$$U^*t = \int [-i \frac{\partial}{\partial E'} \delta(E') \delta(p')] \exp(ipx - iEt) dp dE \quad (45)$$

$$U^*x = \int [i \frac{\partial}{\partial p'} \delta(p') \delta(E')] \exp(ipx - iEt) dp dE, \quad (46)$$

where  $(E', p') = U \circ (E, p)$ . In order to perform the integrals, we need to express  $(E, p)$  as functions of  $(E', p')$ . By definition, we have

$$(E, p) = U^{-1} \circ (E', p') = (h_1(E', p')E', h_2(E', p')p') \quad (47)$$

Hence,

$$\det \frac{\partial(E, p)}{\partial(E', p')} = (h_1 + \dot{h}_1 E')(h_2 + h'_2 p') - h'_1 \dot{h}_2 E' p', \quad (48)$$

where  $h'$  and  $\dot{h}$  are the derivatives with respect to  $p$  and  $E$ , respectively. Thus, Eq. (45) and Eq. (46) can be rewritten as

$$\begin{aligned} U^* x &= \int [i \frac{\partial}{\partial p'} \delta(p') \delta(E')] \exp(ipx - iEt) dp dE \\ &= \int [i \frac{\partial}{\partial p'} \delta(p') \delta(E')] \exp(ih_2 p' x - ih_1 E' t) \det \frac{\partial(E, p)}{\partial(E', p')} dE' dp' \\ &= - \int \delta(p') \delta(E') i \frac{\partial}{\partial p'} \left[ \exp(ih_2 p' x - ih_1 E' t) \det \frac{\partial(E, p)}{\partial(E', p')} \right] dE' dp' \\ &= h_1(0) h_2^2(0) x - i[h'_1(0) h_2(0) + 2h_1(0) h'_2(0)] \end{aligned} \quad (49)$$

$$\begin{aligned} U^* t &= \int [-i \frac{\partial}{\partial E'} \delta(E') \delta(p')] \exp(ipx - iEt) dp dE \\ &= - \int \delta(E') \delta(p') (-i) \frac{\partial}{\partial E'} \left[ \exp(ih_2 p' x - ih_1 E' t) \det \frac{\partial(E, p)}{\partial(E', p')} \right] dE' dp' \\ &= h_1^2(0) h_2(0) t + i[2\dot{h}_1(0) h_2(0) + h_1(0) \dot{h}_2(0)]. \end{aligned} \quad (50)$$

Similarly, we have

$$U^{-1*} x = \frac{x}{h_1(0) h_2^2(0)} + \frac{i}{h_1(0) h_2^2(0)} \left( \frac{h'_1(0)}{h_1(0)} + 2 \frac{h'_2(0)}{h_2(0)} \right) \quad (51)$$

$$U^{-1*} t = \frac{t}{h_1^2(0) h_2(0)} - \frac{i}{h_1^2(0) h_2(0)} \left( 2 \frac{\dot{h}_1(0)}{h_1(0)} + \frac{\dot{h}_2(0)}{h_2(0)} \right) \quad (52)$$

(Note : Here we have used the relation  $\delta(f(x)x) = \frac{1}{f(x)} \delta(x)$ .)

From the above results, we see that  $U^*$  generally is a complex transformation. However, the imaginary parts will vanish if  $h_1$  and  $h_2$  are even functions of  $(E, p)$  and smooth at the origin  $E = p = 0$ . We shall show later in this section that this condition is sufficient to preserve the conservation of electric charge. Therefore, by imposing this condition, we find that  $U^*$  is just a constant dilation, i.e.,

$$U^* x = h_1(0) h_2^2(0) x \quad (53)$$

$$U^* t = h_1^2(0) h_2(0) t. \quad (54)$$



Substituting it into Eq.(11), we get

$$t' = \gamma \left( t - v \frac{h_2(0)}{h_1(0)} x \right) \quad (55)$$

$$x' = \gamma \left( x - v \frac{h_1(0)}{h_2(0)} t \right). \quad (56)$$

As required in [7], transformation (1) should reduce to the ordinary Lorentz transformation for energy scales much smaller than  $E_p$ , which means  $\lim_{E \rightarrow 0} U = 1$ , i.e.,  $h_1(0) = h_2(0) = 1$ , so we find that the induced transformation on coordinate space is just the ordinary Lorentz transformation, which agrees with our result in section 2.1.

Now we explain why we require  $h_1$  and  $h_2$  be even functions. Based on Schützhold and Unruh's result [18], the general form of the map  $U^*$  is

$$\begin{aligned} [U^*\phi](x, t) &= \int B(x, t; \xi, \eta) \phi(\xi, \eta) d\xi d\eta \\ B(x, t; \xi, \eta) &= \int \exp(-ip'\xi + iE'\eta + ipx - iEt) dp dE. \end{aligned} \quad (57)$$

If  $\phi$  is a real scalar field, we wish to see whether  $U^*\phi$  is real too. Based on Eq.(57), the complex conjugate of  $U^*\phi$  is

$$\begin{aligned} \overline{[U^*\phi]}(x, t) &= \overline{\int B(x, t; \xi, \eta) \phi(\xi, \eta) d\xi d\eta} \\ &= \int \int \exp(ip'\xi - iE'\eta - ipx + iEt) \bar{\phi}(\xi, \eta) dp dE d\xi d\eta \\ &= \int \int \exp(if_2 p \xi - if_1 E \eta - ipx + iEt) \phi(\xi, \eta) dp dE d\xi d\eta. \end{aligned} \quad (58)$$

In general,  $\overline{U^*\phi} \neq U^*\phi$ , i.e.,  $U^*\phi$  is not a real function. If  $h_1$  and  $h_2$  are even functions, i.e., invariant under the coordinate transformation  $E \rightarrow -E$  and  $p \rightarrow -p$ , we have

$$\begin{aligned} \overline{U^*\phi} &= \int \int \exp(if_2 p \xi - if_1 E \eta - ipx + iEt) \phi(\xi, \eta) dp dE d\xi d\eta \\ &= \int \int \exp[if_2(-E, -p)(-p)\xi - if_1(-E, -p)(-E)\eta \\ &\quad -i(-p)x + i(-E)t] \phi(\xi, \eta) dp dE d\xi d\eta \\ &= \int \int \exp(-if_2 p \xi + if_1 E \eta + ipx - iEt) \phi(\xi, \eta) dp dE d\xi d\eta \\ &= U^*\phi. \end{aligned} \quad (59)$$

Thus,  $U^*\phi$  is real. If  $U^*\phi$  is not real, it will cause a serious problem. It is well known that a complex field contains electric charges. If the modified Lorentz transformation can not guarantee that  $U^*\phi$  is real, different observers will have different views on whether a particle is charged, which is in contradiction with the conservation of electric charge. However, we only show that the even function requirement is a sufficient condition for  $U^*\phi$  being real. It may not be necessary.

### 2.3 Coordinate transformation from field equation

As outlined in [7], the derivatives in a field equation should transform as momentum. One can construct the modified scalar field equation by the replacement

$$p_a \rightarrow i\partial_a \quad (60)$$

applied to the dispersion relations. Thus, from Eq. (10), we have the modified Klein-Gordon equation[10]

$$\eta^{ab} \frac{\partial_a}{1 - il_p \partial_0} \frac{\partial_b}{1 - il_p \partial_0} \phi(x) = 0, \quad (61)$$

which has plane wave solutions  $\phi = Ae^{-ipx}$  with  $p_a$  satisfying the dispersion relation Eq. (10). A basic requirement for a field equation is that it must be covariant under coordinate transformation between inertial frames. Conversely, this requirement can be used to determine possible coordinate transformations that keep the field equation (61) invariant. The explicit expression of Eq. (61) is obtained by Taylor expansion

$$\eta^{ab} \partial_a (1 - il_p \partial_0 \dots) \partial_b (1 - il_p \partial_0 \dots) \phi(x) = 0. \quad (62)$$

It is not difficult to see that in four dimensions, pure spatial rotations will make Eq. (62) invariant. Now we show that spatial rotations actually are the only linear transformations on position space that make Eq. (62) invariant. Without loss of generality, we still consider 1 + 1 spacetimes and Eq. (62) becomes

$$-\partial_t^2 \phi(x) + \partial_x^2 \phi(x) - 2il_p \partial_t^3 \phi(x) + 2il_p \partial_x^2 \partial_t \phi(x) + \dots = 0. \quad (63)$$

Consider coordinates  $(t', x')$ . From the chain rule, we have

$$\partial_t = \frac{\partial t'}{\partial t} \partial'_t + \frac{\partial x'}{\partial t} \partial'_x \quad (64)$$

$$\partial_x = \frac{\partial t'}{\partial x} \partial'_t + \frac{\partial x'}{\partial x} \partial'_x. \quad (65)$$

Substituting Eq. (64) and Eq. (65) into Eq. (63), we get

$$\begin{aligned}
& -\partial_t^2 \phi(x) + \partial_x^2 \phi(x) - 2il_p \partial_t^3 \phi(x) + 2il_p \partial_x^2 \partial_t \phi(x) + \dots \\
= & \left[ -\left(\frac{\partial x'}{\partial t}\right)^2 + \left(\frac{\partial x'}{\partial x}\right)^2 + \dots \right] \phi'' + \left[ -\left(\frac{\partial t'}{\partial t}\right)^2 + \left(\frac{\partial t'}{\partial x}\right)^2 + \dots \right] \ddot{\phi} + \left[ -\frac{\partial x'}{\partial t} \frac{\partial}{\partial x} \left(\frac{\partial x'}{\partial t}\right) + \dots \right] \phi' \\
& + \left[ -\frac{\partial x'}{\partial t} \frac{\partial}{\partial x} \left(\frac{\partial t'}{\partial t}\right) + \dots \right] \dot{\phi} + \left[ -2\frac{\partial t'}{\partial t} \frac{\partial x'}{\partial t} + 2\frac{\partial t'}{\partial x} \frac{\partial x'}{\partial x} + \dots \right] \dot{\phi}' \\
& - 2il_p \left[ 3 \left(\frac{\partial x'}{\partial t}\right)^2 \frac{\partial}{\partial x} \left(\frac{\partial x'}{\partial t}\right) + \dots \right] \phi'' - 2il_p \left[ 3 \left(\frac{\partial t'}{\partial x}\right)^2 \frac{\partial}{\partial t'} \left(\frac{\partial t'}{\partial t}\right) + \dots \right] \ddot{\phi} \\
& - 2il_p \left[ \left(\frac{\partial t'}{\partial t}\right)^3 - \frac{\partial t'}{\partial t} \left(\frac{\partial t'}{\partial x}\right)^2 + \dots \right] \partial_t^3 \phi \\
& - 2il_p \left[ 3 \frac{\partial t'}{\partial t} \left(\frac{\partial x'}{\partial t}\right)^2 - 2 \frac{\partial t'}{\partial x} \frac{\partial x'}{\partial t} \frac{\partial x'}{\partial x} - \frac{\partial t'}{\partial t} \left(\frac{\partial x'}{\partial x}\right)^2 + \dots \right] \partial_x^2 \partial_t \phi \\
& - 2il_p \left[ 3 \left(\frac{\partial t'}{\partial t}\right)^2 \frac{\partial x'}{\partial t} - \left(\frac{\partial t'}{\partial x}\right)^2 \frac{\partial x'}{\partial t} - 2 \frac{\partial t'}{\partial t} \frac{\partial t'}{\partial x} \frac{\partial x'}{\partial x} + \dots \right] \partial_t^2 \partial_x \phi + \dots \tag{66}
\end{aligned}$$

Covariance means that the corresponding coefficients in Eq. (63) and Eq. (66) be equal. For linear transformation,  $\frac{\partial t'}{\partial t}$ ,  $\frac{\partial x'}{\partial t}$ ,  $\frac{\partial t'}{\partial x}$ ,  $\frac{\partial x'}{\partial x}$  are constants. Then, the coefficients of  $\phi''$ ,  $\ddot{\phi}$  and  $\dot{\phi}'$  yield <sup>1</sup>

$$\left(\frac{\partial t'}{\partial t}\right)^2 - \left(\frac{\partial t'}{\partial x}\right)^2 = 1 \tag{67}$$

$$\left(\frac{\partial x'}{\partial x}\right)^2 - \left(\frac{\partial x'}{\partial t}\right)^2 = 1 \tag{68}$$

$$-2\frac{\partial t'}{\partial t} \frac{\partial x'}{\partial t} + 2\frac{\partial t'}{\partial x} \frac{\partial x'}{\partial x} = 0, \tag{69}$$

These three equations just give the Lorentz transformation. However, by comparing the coefficients of  $\partial_t^3 \phi$ , we obtain

$$\frac{\partial t'}{\partial t} = 1, \tag{70}$$

which means that no boost is allowed for the linear transformation.

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<sup>1</sup>The skipped terms in the coefficients of Eq. (66) (denoted by “...” ) involve derivatives of  $\frac{\partial t'}{\partial t}$ ,  $\frac{\partial x'}{\partial t}$ ,  $\frac{\partial t'}{\partial x}$ ,  $\frac{\partial x'}{\partial x}$ , which vanish in linear transformations.

Kimberly, *et al.*[10] argues that since there are plane wave solutions, the linear contraction  $p_a dx^a$  must be invariant. Therefore, the corresponding transformation on position space is

$$dt' = \gamma(dt - vdx)[1 + (\gamma - 1)l_p E - \gamma l_p v p_x] \quad (71)$$

$$dx' = \gamma(dx - vdt)[1 + (\gamma - 1)l_p E - \gamma l_p v p_x]. \quad (72)$$

Although the above relations, originally shown in [10], are for infinitesimal separations  $dt, dx$ , they should also hold for finite separations. Hence, we have an energy-dependent linear transformation on position space. However, this contradicts our conclusion that no boost is allowed for linear transformations except for the case  $v = 0$ .

### 3 Conclusions

We present three distinct methods to translate the behavior in momentum space into position space. We first identify the boost velocity with the real velocity measured by an inertial observer. Then we show that any non-linear transformation in momentum space always leads to the usual Lorentz transformation in position space. By applying Fourier transform to a scalar field, we also obtain the same result. However, some additional requirements have to be imposed. From the covariance of the modified Klein-Gordon equation, we show that only pure rotations are permitted among linear transformations and the condition that  $p_a dx^a$  remains an invariant scalar is not compatible with the covariance of the field equation.

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